Population Projection Model using Exponential Growth Function with a Birth and Death Diffusion Growth Rate Processes

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Abstract

One of the important functions of the demographers is to provide information on the future trend of population projection, which is important to plan for human activities. Today, demographers are interested in describing phenomena in theoretical models involving population structure by considering the stochastic analogs of classical differences and differential equations. In this paper, a description of a population projection model is derived using a growth rate follow a birth and death diffusion process. The mean and the variance as well as the predicted and the simulated sample path of such a population projection process are also obtained. Numerical example for the case of the birth and death diffusion growth rate as well as the case of the constant growth rate are considered.

Keywords: Population Projection Model, Birth-Death Diffusion Process, Growth Rate Process, Predicted Population, Sample Path

1. Introduction and Background

Malthus while giving the first qualitative formulation of population growth in 1789 observed that the increase of population follows a Geometric Progression in contrast to its means of subsistence with tend to grow in arithmetical progression. Under condition of unlimited resources, Malthusian Geometric low can be expressed as

$$\frac{dP(t)}{dt} = rP(t)$$

(1)

Solving the differential equation in equation (1), we get

$$P(t) = P(0)e^{rt}$$

(2)

Where, $P(t)$ is the population at any time $t$ and $r$ is the constant of proportionality. Note that $P(0)$ refers to the initial size of the population (c.f. Biswas (1988)).
Malthus did not take into account the fact that in any given environment the growth of population may stop due to the density of population, which the environment can sustain. However, in 1838, Verhulst took account of this limitation of Malthus. He postulated that the rate of the population growth was jointly proportional to the existing population (c.f. Biswas (1988)).

It is known that neither will \( r \) be constant over the interval 0 to \( t \) nor can any particular value of it represent the future. In other words, several values could be assigned to \( P(t) \) depending upon the assumptions and the models used. Thus, instead of using one parameter as in (2), we can choose models of several parameters according to the contribution of their individual uncertainty.

Numerous researchers have worked on studying various problems related directly and indirectly to population projection from different points of view. For example, Cohen (1995) mentioned that the human carrying capacity is dynamic and uncertain. Therefore, he utilized a simple mathematical modeling relationship projection to capture the fast exponential population growth followed by slower population growth rate. Moreover, this projection model stood between both human population growth and carrying capacity.

Tayman et. al. (1999) studied the relationship between the population size and the accuracy of the projections on a group of county areas in USA and found that there is an inverse relationship between them. Also, he found the specific age extrapolation tends to decrease in speed relatively to past performance when the mortality rate is considered. This result, which has been found in (Lee & Carter 1992, Wilmoth 1998, Tuljapurkar, Li and Boe 2000). On the other hand, the mortality rate is increasing in younger ages more than their past occurrence in years 1999 and 2003 by the Technical Panel.

According to Wilkinson (2003), it is possible understand and examine the role of the population projection regarding development, accuracy and why/how they can be applicable to so many contributed variables. Furthermore, the variations of the current range of projections can be examined to highlight on the amount of challenges facing both projection modeling and outputs.

In this paper, we present a new population projection model using a growth rate follow a birth and death diffusion process. The mean and the variance as well as the predicted and the simulated sample path of such a population projection model are obtained. Numerical example for the case of the birth and death diffusion growth rate as well as the case of the constant growth rate are considered.

The objective of this research is to identify critical knowledge types required by demographers of population projection techniques through building a population projection model that has never been examined before as far as I know. The results should be very useful and will benefit the demographers and others to study the behavior of populations through different applications.

2. Population Growth Model Using a Birth and Death Diffusion Growth Rate Process

Let \( P(t) \) be the population size at time \( t \) with a population growth rate \( r(t) \) at time \( t \). Then, the population growth model will be given by

\[
\frac{dP(t)}{dt} = r(t)P(t)
\]  

Solving the above differential equation in equation (3), we get

\[
P(t) = P(0)e^{\int_0^t r(t)dt}
\]  

Now, assuming

\[
R(t) = \int_0^t r(t)dt
\]  

Substituting equation (5) in equation (4), we get the
\[ P(t) = P(0)e^{R(t)} \]  
(6)

Where, \( P(t) \) is the population at any time \( t \) and \( r(t) \) is the growth rate. Moreover, \( P(0) \) refers to the initial size of the population. Note that when \( r(t) = r \), then equation (6) gives the same model given in equation (2).

Now, consider \( \{r(t); t \geq 0\} \) the birth and death diffusion growth rate process, in which the diffusion coefficient \( a \) and the drift coefficient \( b \) are both, proportional to \( r(t) \) at time \( t \). Then \( \{r(t); t \geq 0\} \) is a Markov process with State Space \( S = [0, \infty) \) and can be regarded as a solution of the stochastic differential equation

\[ dr(t) = br(t)dt + ar(t)dW(t) \]  
(7)

Here \( \{W(t)\} \) is a Wiener process with mean zero and variance \( \sigma^2 t \).

Now from equation (7) we get,

\[ \frac{dL(t)}{r(t)} = bdt + adW(t) \]  
(8)

Thus, the solution of the stochastic differential equation in (8) is given by

\[ r(t) = r(0)\exp\{bt + aW(t)\} \]  
(9)

where \( r(0) \) is the initial growth rate at time zero.

Now, solving \( R(t) \) in equation (5), using Taylor and Karlin (1984, pp. 177) and Al-Eideh and Al-Hussainan (2002) and after some algebraic manipulations, it is easily shown that

\[ R(t) = \int_0^t r(s)ds = \int_0^t r(0)\exp\{bs + aW(s)\}ds \]

\[ = \frac{2(1-b)}{2a + a^2 - b^2} r(0)\exp\{bt + aW(t)\} \]  
(10)

Therefore, the population size \( P(t) \) at time \( t \) using the birth and death diffusion growth rate process defined in equation (6) is then given by

\[ P(t) = P(0)\exp\left\{ \frac{2(1-b)}{2a + a^2 - b^2} r(0)\exp\{bt + aW(t)\} \right\} \]  
(11)

where \( P(0) \) and \( r(0) \) are the initial population and the initial growth rate at time zero.

3. Mean and Variance of the Population Model \( P(t) \) using the Birth and Death Diffusion Growth Rate Process

In this section, the mean and the variance for the population model \( P(t) \) using the birth and death growth rate process will be obtained.

Let \( M_t(t) = E[P(t)] \) and \( V(t) = V[P(t)] \) be the mean and the variance of \( P(t) \) respectively.

Using the results of finding the moment approximation of a birth and death diffusion process (cf. Al-Eideh (2001)), it is easily shown that

\[ E[P(t)] = P(0)\exp\left\{ \frac{2(1-b)}{2a + a^2 - b^2} r(0)\exp\left\{ \left[ b + \frac{1}{2} a\sigma^2 \right]t \right\} \right\} \]  
(12)

And

\[ E[P^2(t)] = (P(0))^2 \exp\left\{ 2 \left( \frac{2(1-b)}{2a + a^2 - b^2} \right)^2 r(0)\exp\{2b + 2a^2 \sigma^2 \} \right\} \]  
(13)

Therefore, the variance of \( P(t) \) is then given by
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\[
V[P(t)] = (P(0))^2 \exp\left\{2\left(\frac{2(1-b)}{2a + a^2 - b^2}\right)^2 r(0) \exp\left\{2b + 2a^2 \sigma^2\right\}\left\{e^{a^2 \sigma^2} - 1\right\}\right\}
\]

where \(P(0)\) and \(r(0)\) are the initial population and the initial growth rate at time zero.

4. Predicted And Simulated Population Model \(P(t)\)

In this section, we will obtain the predicted and the simulated sample path of the Population process \(P(t)\) using the birth and death diffusion process.

Assuming \(M_1(t_n - t_{n-1})\) be the one-step predicted model of \(P(t)\), then \(M_1(t_n - t_{n-1})\) can be written as

\[
M_1(t_n - t_{n-1}) = P(0) \exp\left\{\frac{2(1-b)}{2a + a^2 - b^2} r(0) \exp\left\{\left(b + \frac{1}{2} a \sigma^2\right)t_n - t_{n-1}\right\}\right\}
\]

For simulation of the population process, \(P(t)\) we used the following discrete approximation.

For integer values \(k = 1, 2, 3, \ldots\), and \(n = 1, 2, 3, \ldots\), the growth rate diffusion process \(r(t)\) can be simulated by

\[
r_n^*\left(\frac{k+1}{n}\right) = r_n^*\left(\frac{k}{n}\right) + \left(\frac{2(1-b)}{2a + a^2 - b^2}\right) \left(\frac{br_n^*\left(\frac{k}{n}\right)}{n} + \frac{a}{n} r_n^*\left(\frac{k}{n}\right)\right) Z_{k+1}
\]

where \(\{Z(k)\}\) is an independent sequence of standard normal random variables

For each set of positive integers \(k, t_1, \ldots, t_k\), the sequence of random vectors \((r_n^*(t_1), \ldots, r_n^*(t_k))\)' converges in distribution to \((r_n^*(t_1), \ldots, r_n^*(t_k))\)'.

Thus, the simulated population model \(P(t)\) is given by

\[
P_n^*\left(\frac{k+1}{n}\right) = P_n^*\left(\frac{k}{n}\right) \exp\left\{r_n^*\left(\frac{k+1}{n}\right)\right\}
\]

where \(r_n^*\left(\frac{k+1}{n}\right)\) in defined in equation (16).

Note that for each set of positive integers \(k\), the sequence of random vectors \(P_n^*(1), \ldots, P_n^*(k)\)' converges in distribution to \(P(1), \ldots, P(k)\)'.

5. Numerical Example

Consider as an example the following sample paths of the above model \(P(t)\) in section (4) that represents the annual Population of an anonymous country (in \(10^3\)) when \(P(0) = 2133\), \(r(0) = 0.042\), \(b = 0.042\), \(a = 0.1\), \(r = 0.042\) and \(n = 26\).

Now, we consider the sample path to the Population projection model \(P(t)\) using the birth and death diffusion growth rate process \(r(t)\) as obtained in equation (11) and compare the results with the Old Population projection model defined in equation (2) with constant growth rate \(r(t) = r\) (Note that this model is denoted by \(OP(t)\) in the figures). Figure 1 and Figure 2 represent these plots for \(r(t)\) and \(P(t)\) respectively.
Looking to the above figures, we see the difference between these figures, also the difference between the new and the old projection models is noted in the birth and death growth rate process and in the constant growth rate. In addition, the growth rates and finally this difference affects the new population projection model in equation (11). Any way the figures are reasonable and suggested to be used in the modeling purposes for population projections.

6. Conclusion
In conclusion, this study provides a methodology for studying the behavior of the population projection. More specifically, this study departs from the traditional before and after regression techniques and the time series analysis, and developed a population model that explicitly accounts for the variations and volatilities in population size using a population projection with birth and death diffusion growth rate process. In addition, some inference problems could be done for this model.
References


