Retraction of Fuzzy Hypercylinder

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Abstract

Our aim in the present paper is to introduce and study new connection between fuzzy retraction, fuzzy folding, and fuzzy deformation retract of fuzzy n-dimensional cylinder. Types of fuzzy foldings and fuzzy deformation retracts of fuzzy n-dimensional cylinder are discussed. Types of minimal fuzzy retraction of fuzzy n-dimensional cylinder are obtained. The fuzzy foldings of fuzzy n-dimensional cylinder are deduced. The relation between the fuzzy deformation retract of fuzzy n-dimensional cylinder and the fuzzy deformation retract of fuzzy tangent space are achieved. Some commutative diagrams are obtained. Some applications are presented.

Keywords: Fuzzy retraction, Fuzzy deformation retract, Fuzzy folding, Fuzzy hypercylinder

Mathematics Subject Classification: 51H10, 57N10

Introduction and Background

There are many diverse applications of certain phenomena for which it is impossible to get relevant data. It may not be possible to measure essential parameters of a process such as the temperature inside molten glass or the homogeneity of a mixture inside some tanks. The required measurement scale may not exist at all, such as in the case of evaluation of offensive smells, evaluating the taste of foods or medical diagnoses by touching [1-15]. The aim of the present paper is to describe the above phenomena geometrically, specifically concerned with the study of the new types of fuzzy retractions, fuzzy folding and fuzzy deformation retract of fuzzy hypercylinder \( \mathbb{L}^n \subset \mathbb{R}^{n+2} \).

A fuzzy manifold is manifold which has a physical character. This character is represented by the density function \( \mu \), where \( \mu \in [0,1] \) [7, 8].
A fuzzy subset \((A, \mu)\) of a fuzzy manifold \((\tilde{M}, \mu)\) is called a fuzzy retraction of \((\tilde{M}, \mu)\) if there exist a continuous map \(\phi: (\tilde{M}, \mu) \rightarrow (A, \mu)\) such that \(\phi(a, \mu(a)) = (a, \mu(a)), \forall a \in A, \mu \in [0,1]\) [2,3,16-25].

A fuzzy subset \((M, \mu)\) of a fuzzy manifold \((\tilde{M}, \mu)\) is called a fuzzy deformation retract if there exists a fuzzy retraction \(\phi: (\tilde{M}, \mu) \rightarrow (M, \mu)\) and a fuzzy homotopy \(\Phi: (\tilde{M}, \mu) \times I \rightarrow (\tilde{M}, \mu)\) [4,5,6,16-25] such that

\[
\Phi((a, \mu), 0) = (a, \mu),
\]

\[
\Phi((a, \mu), 1) = \phi(x, \mu)
\]

\(\forall x \in M, \mu \in [0,1].\) Where \(\phi(x, \mu)\) is the retraction mentioned above.

A map \(\tilde{M}: \tilde{L}^n \rightarrow \tilde{L}^n\) is said to be an isometric folding of fuzzy hypercylinder \(\tilde{L}^n\) into itself iff for any piecewise fuzzy geodesic path \(\gamma: J \rightarrow \tilde{L}^n\) the induced path \(\tilde{M} \circ \gamma: J \rightarrow \tilde{L}^n\) is a piecewise fuzzy geodesic and of the same length as \(\gamma\), where \(J = [0,1]\). If \(\tilde{M}\) does not preserve lengths, then \(\tilde{M}\) is a topological folding of fuzzy hypercylinder \(\tilde{L}^n\) [7,8,9].

The isofuzzy folding of \(\cup \tilde{M}_i\) is a folding \(\tilde{M}: \cup \tilde{M}_i \rightarrow \cup \tilde{M}_i\) such that \(\tilde{M}(\tilde{M}_i) = \tilde{M}\) and any \(\tilde{M}_i\) belong to the upper hypermanifolds \(\exists \tilde{M}_j\) down \(\tilde{M}\) such that \(\mu_i = \mu_j\) for every corresponding points i.e. \(\mu(a_i) = \mu(a_j)\) [10,11]. See Fig. (1).

**Figure 1:**

\[\begin{align*}
\tilde{M}_i & \cup a_i \\
\tilde{M} & \cup a_j
\end{align*}\]

**Main Results**

**Theorem 1.** The fuzzy retractions of the fuzzy hypercylinder \(\tilde{L}^n \subset \tilde{R}^{n+2}\) induces two chains of fuzzy retractions of the two fuzzy systems of fuzzy hypercylinders \(\cup \tilde{L}_i\), and \(\cup \tilde{L}_j\).

**Proof.** Let \(\tilde{L}^n \subset \tilde{R}^{n+2}\) be a fuzzy hypercylinder and \(\beta_i \in \tilde{L}^n\), where \(\beta_i\) is homeomorphic to \(S_i\), then there are induced nested 2- chains of \(n\)- pure fuzzy hypercylinder \(\cup \tilde{L}_i\) and \(\cup \tilde{L}_j\). The fuzzy parametric equation of the fuzzy hypercylinder \(\tilde{L}^n\) is presented by

\[\tilde{x}_i = r \prod_{k=1}^{n-1} \sin \theta_k(\eta), \quad \tilde{x}_i = r \cos \theta_{i-1}(\eta) \prod_{k=1}^{n-1} \sin \theta_k(\eta), \quad 1 < i < n + 1, \quad \tilde{x}_{n+1} = r \cos \theta_n(\eta), \quad x_{n+2} = \theta_{n+1}(\eta), \quad \text{where } \eta \in [0,1].\]

(I) Now, if \(r_1 < r, i=1,2,3,\ldots\infty\). On \(\cup \tilde{L}_i\), the fuzzy retraction \(\tilde{R}_1, \tilde{R}_1 : (\tilde{L}^n - \{\beta_{i1}\} \rightarrow \tilde{L}^{n-1}\), this retraction induces two fuzzy retractions \(\tilde{R}_1, \tilde{R}_1 : (\tilde{L}^n - \{\beta_{i1}\} \rightarrow \tilde{L}^{n-1}\), where the fuzzy parametric equation of the fuzzy hypercylinder \(\tilde{L}^{n-1}\) is presented by

\[\tilde{R}_1 = (\tilde{x}_i = r \prod_{k=1}^{n-1} \sin \theta_k(\eta), \quad \tilde{x}_i = r \cos \theta_{i-1}(\eta) \prod_{k=1}^{n-1} \sin \theta_k(\eta), \quad 1 < i < n, \quad \tilde{x}_{n+1} = r \cos \theta_n(\eta), \quad x_{n+2} = \theta_{n+1}(\eta), \quad \text{where } \eta \in [0,1], r_1 < r.\]

Also, the fuzzy retraction \(\tilde{R}_2 : (\tilde{L}^{n-1} - \{\beta_{i2}\} \rightarrow \tilde{L}^{n-2}\).
\[
\bar{R}_2 = (\bar{x}_i = r \prod_{k=1}^{n-1} \sin \theta_k(\eta), \bar{x}_{n+1} = r \cos \theta_{n-1}(\eta) \prod_{k=1}^{n-1} \sin \theta_k(\eta)), \quad 1 < i < n - 1, \quad \bar{x}_{n+1} = r \cos \theta_n(\eta), \quad \bar{x}_{n+2} = \theta_{n+1}(\eta), \text{ where } \eta \in [0,1], \quad r_2 < r_1 < r. \]

This retraction induces two fuzzy retractions \( \bar{R}_2 : (\bar{L}_i^n - \beta_{i2}) \to \bar{L}_i^{n-2} \), \( \bar{R}_2 : (\bar{L}_i^n - \beta_{i2}) \to \bar{L}_i^{n-2} \). Also, the fuzzy retraction \( \bar{R}_3 : (\bar{L}_i^n - \beta_{i3}) \to \bar{L}_i^{n-3} \). This retraction induces two fuzzy retractions \( \bar{R}_3 : (\bar{L}_i^n - \beta_{i3}) \to \bar{L}_i^{n-3} \), \( \bar{R}_3 : (\bar{L}_i^n - \beta_{i3}) \to \bar{L}_i^{n-3} \), where the fuzzy parametric equation of the fuzzy hypercylinder \( \bar{L}_i^{n-3} \) is presented as

\[
\bar{R}_3 = (\bar{x}_i = r \prod_{k=1}^{n-1} \sin \theta_k(\eta), \bar{x}_{n+1} = r \cos \theta_{n-1}(\eta) \prod_{k=1}^{n-1} \sin \theta_k(\eta), \quad 1 < i < n - 2, \quad \bar{x}_{n+1} = r \cos \theta_n(\eta), \quad \bar{x}_{n+2} = \theta_{n+1}(\eta), \text{ where } \eta \in [0,1], \quad r_3 < r_2 < r_1 < r, \ldots.
\]

Consequently, the fuzzy retraction \( \bar{R}_{n-1} \), \( \bar{R}_{n-1} : (\bar{L}_i^2 - \beta_{i(n-1)}) \to \bar{L}_i^1 \), induces two fuzzy retractions \( \bar{R}_{n-1} : (\bar{L}_i^2 - \beta_{i(n-1)}) \to \bar{L}_i^1 \), \( \bar{R}_{n-1} : (\bar{L}_i^2 - \beta_{i(n-1)}) \to \bar{L}_i^1 \), where the fuzzy parametric equation of the fuzzy hypercylinder \( \bar{L}_i^2 \) is given by

\[
\bar{R}_{n-1} = \{ r \cos \theta_1(\eta) \cos \theta_2(\eta), \quad r \sin \theta_1(\eta) \cos \theta_2(\eta), \quad r \sin \theta_2(\eta), \quad \theta_3(\eta) \},
\]

\( r_{n-1} < r_{n-2} < r_{n-3} < \cdots < r_3 < r_2 < r_1 < r. \)

The fuzzy retraction \( \bar{R}_n, \bar{R}_n : (\bar{L}_i^1 - \beta_{i(n)}) \to \bar{S}_i^1 \), induces two fuzzy retractions \( \bar{R}_n : (\bar{L}_i^1 - \beta_{i(n)}) \to \bar{S}_i^1 \), \( \bar{R}_n : (\bar{L}_i^1 - \beta_{i(n)}) \to \bar{S}_i^1 \), where the fuzzy parametric equation of the fuzzy hypercylinder \( \bar{L}_i^1 \) is given by

\[
\bar{R}_n = \{ r \cos \theta_1(\eta), \quad r \sin \theta_1(\eta) \cos \theta_2(\eta), \quad \theta_2(\eta) \}, \quad r_n < r_{n-1} < r_{n-2} < r_{n-3} < \cdots < r_3 < r_2 < r_1 < r.
\]

The fuzzy retraction \( \bar{R}_{n+1}, \bar{R}_{n+1} : (\bar{S}_i^1 - \beta_{i(n+1)}) \to \bar{S}_i^0 \), induces two fuzzy retractions \( \bar{R}_{n+1} : (\bar{S}_i^1 - \beta_{i(n+1)}) \to \bar{S}_i^0 \), \( \bar{R}_{n+1} : (\bar{S}_i^1 - \beta_{i(n+1)}) \to \bar{S}_i^0 \), where the fuzzy parametric equation of the fuzzy circle \( \bar{S}_i^1 \) is given by \( \bar{R}_{n+1} = \{ r \cos \theta_1(\eta), \quad r \sin \theta_1(\eta), \quad 0 \}, \quad r_n < r < r_{n-1} < r_{n-2} < r_{n-3} < \cdots < r_3 < r_2 < r_1 < r. \)

(II) If \( r_1 > r \), \( i = 1, 2, 3, \ldots \). On \( \bigcup \bar{L}_i^n \). We have the same results as in(I).

**Corollary 1.** Under the condition \( \bar{x}_{n+2} = 0 \), the fuzzy retractions of the fuzzy hyperspheres \( \bar{S}_i^n \subset \bar{R}^{n+1} \) induces two chains of fuzzy retractions of the two fuzzy systems of fuzzy hyperspheres \( \bigcup \bar{S}_i^n \) and \( \bigcup \bar{S}_i^n \).

**Proof.** From Theorem (1) the proof is clear and also see Fig. (2)

**Figure 2:**

![Figure 2](image)

**Theorem 2.** The folding of the fuzzy hypercylinder \( \bar{L}_i^n \subset \bar{R}^{n+2} \) into itself induces two chains of fuzzy folding \( \bar{L}_i^n \) and \( \bar{L}_i^n \) which is a type of fuzzy retractions.
Proof. Assume \( \mathcal{F}_1: \bar{L}^n \to \bar{L}^n \) be a fuzzy folding from \( \bar{L}^n \) into \( \bar{L}^n \) such that \( \mathcal{F}_1(\bar{L}^n) \neq \bar{L}^n \). This folding induces 2-chains of fuzzy folding \( \mathcal{F}_1: \bar{L}^n \to \bar{L}^n \) such that \( \mathcal{F}_1(\bar{L}^n) \neq (\bar{L}^n) \) and \( \mathcal{F}_1: \bar{L}^n \to \bar{L}^n \) such that \( \mathcal{F}_1(\bar{L}^n) \neq (\bar{L}^n) \), where \( \dim \mathcal{F}_1 = \dim \bar{F}_1 = \dim \mathcal{F}_1 = \dim \bar{R}_1 \). Then the fuzzy folding not conside with the fuzzy retraction. Also, let \( \mathcal{F}_2: \mathcal{F}_1(\bar{L}^n) \to \mathcal{F}_1(\bar{L}^n) \), this folding induces \( \mathcal{F}_2 \) and \( \bar{F}_2 \) where \( \bar{F}_2 : \mathcal{F}_1(\bar{L}^n) \to \mathcal{F}_1(\bar{L}^n) \), \( \bar{F}_3 : \bar{F}_2 (\mathcal{F}_1(\bar{L}^n)) \to \bar{F}_2 (\mathcal{F}_1(\bar{L}^n)) \), \( \bar{F}_3 : \bar{F}_2 (\mathcal{F}_1(\bar{L}^n)) \to \bar{F}_2 (\mathcal{F}_1(\bar{L}^n)) \), \( \bar{F}_3 : \bar{F}_2 (\mathcal{F}_1(\bar{L}^n)) \to \bar{F}_2 (\mathcal{F}_1(\bar{L}^n)) \), \( \bar{F}_3 : \bar{F}_2 (\mathcal{F}_1(\bar{L}^n)) \to \bar{F}_2 (\mathcal{F}_1(\bar{L}^n)) \), ... \( \bar{F}_n : \bar{F}_n-1 (\bar{F}_n-2 (\bar{F}_n-3 \cdots \bar{F}_2 (\mathcal{F}_1(\bar{L}^n)) \cdots)) \to \bar{F}_n-1 (\bar{F}_n-2 (\bar{F}_n-3 \cdots \bar{F}_2 (\mathcal{F}_1(\bar{L}^n)) \cdots)). \)

Then the fuzzy folding induces two chains of fuzzy upper foldings and fuzzy lower folding
\[
\bar{F}_n : \bar{F}_n-1 (\bar{F}_n-2 (\bar{F}_n-3 \cdots \bar{F}_2 (\mathcal{F}_1(\bar{L}^n)) \cdots)) \to \bar{F}_n-1 (\bar{F}_n-2 (\bar{F}_n-3 \cdots \bar{F}_2 (\mathcal{F}_1(\bar{L}^n)) \cdots)).
\]

Corollary 2. Under the condition \( \bar{F}_{n+2} = 0 \), the folding of the fuzzy hyperspheres \( \bar{S}_n \subset \bar{R}^{n+1} \) into itself induces two chains of folding \( \bar{S}_i \) and \( \bar{S}_i \) which is a type of fuzzy retraction.

Theorem 3. If the fuzzy retraction of the fuzzy hypercylinder \( \bar{L}^n \subset \bar{R}^{n+2} \) is \( \bar{R}: (\bar{L}^n - \{\beta_n\}) \to \bar{L}^{n-1} \) and the fuzzy folding of \( (\bar{L}^n - \{\beta_n\}) \) into itself is \( \bar{F}: (\bar{L}^n - \{\beta_n\}) \to (\bar{L}^n - \{\beta_n\}) \), then there are induces 2-chains of fuzzy rections and foldings such that the following diagram is commutatives.

Proof. Let the fuzzy retraction of \( \bar{L}^n \) is defined by \( \bar{R}_1: (\bar{L}^n - \{\beta_n\}) \to (\bar{L}^n - \{\beta_n\}) \) and the fuzzy folding of \( \bar{L}^n \) and \( \bar{L}^{n-1} \) are given by \( \bar{F}_1 : (\bar{L}^n - \{\beta_n\}) \to (\bar{L}^n - \{\beta_n\}) \), \( \bar{F}_2 : (\bar{L}^n - \{\beta_n\}) \to (\bar{L}^n - \{\beta_n\}) \). Also, \( \bar{R}_2 : \bar{F}_1 (\bar{L}^n - \{\beta_n\}) \to \bar{L}^{n-1} \). Then, there are induced 2-chains of fuzzy rections and foldings are given by
\[
\bar{R}_1 : (\bar{L}^n - \{\beta_n\}) \to (\bar{L}^n - \{\beta_n\}), \bar{F}_1 : (\bar{L}^n - \{\beta_n\}) \to (\bar{L}^n - \{\beta_n\}),
\]
\[
\bar{F}_1 : (\bar{L}^n - \{\beta_n\}) \to (\bar{L}^n - \{\beta_n\}), \bar{F}_2 : \bar{F}_1 (\bar{L}^n - \{\beta_n\}) \to \bar{L}^{n-1} \rightarrow \bar{L}^{n-1}.
\]

Hence, the following diagrams are commutative
\[
\begin{array}{c}
\{\bar{L}^n - \{\beta_n\}\} \xrightarrow{\bar{F}_1} \bar{L}^{n-1} \\
\downarrow \bar{F}_2 \quad \downarrow \bar{F}_1 \\
\{\bar{L}^n - \{\beta_n\}\} \xrightarrow{\bar{F}_1} \bar{L}^{n-1}
\end{array}
\]
\[
\begin{array}{c}
\{\bar{L}^n - \{\beta_n\}\} \xrightarrow{\bar{F}_1} \bar{L}^{n-1} \\
\downarrow \bar{F}_2 \quad \downarrow \bar{F}_1 \\
\{\bar{L}^n - \{\beta_n\}\} \xrightarrow{\bar{F}_1} \bar{L}^{n-1}
\end{array}
\]

Moreover the generalization of theorem represented by the following chins
\[
\bar{F}_1 = \bar{F}_1 \circ \bar{F}_1 \circ \bar{F}_1, \bar{R}_1 = \bar{R}_1 \circ \bar{R}_1 \circ \bar{R}_1, \bar{F}_1 = \bar{F}_1 \circ \bar{F}_1 \circ \bar{F}_1, \bar{R}_1 = \bar{R}_1 \circ \bar{R}_1 \circ \bar{R}_1, i=1,2,...,n.
\]
Theorem 4. Let $\bar{L}^n$ be a fuzzy hypercylinder in a fuzzy Euclidean space $\bar{R}^{n+2}$, $\bar{\mathcal{S}}_i : \bar{L}^n \to \bar{L}^n$. Then $\bar{\mathcal{S}}_{i+1} \circ \bar{r}_i = \bar{\mathcal{S}}_i \circ \bar{r}_i$.

Proof: Consider the fuzzy retractions $(\bar{L}^{n-1} - \{\beta_n\})$, $(\bar{L}^{n-2} - \{\beta_n\})$, $(\bar{L}^{n-3} - \{\beta_n\})$, $(\bar{L}^3 - \{\beta_n\})$, $(\bar{L}^2 - \{\beta_n\})$, $(\bar{L}^1 - \{\beta_n\})$ in a fuzzy hypercylinder $\bar{L}^n$ and $\bar{\mathcal{S}}_0 : \bar{L}^n \to \bar{L}^n$, $\bar{\mathcal{S}}_1 : \bar{L}^{n-1} \to \bar{L}^{n-1}$, ..., $\bar{\mathcal{S}}_{n-1} : \bar{L}^1 \to \bar{L}^1$ be a fuzzy foldings of $\bar{L}^n$. Then we get the following chain.

This is the general commutative diagram such that $\bar{\mathcal{S}}_{i+1} \circ \bar{r}_i = \bar{\mathcal{S}}_i \circ \bar{r}_i$.

Theorem 5. Let the fuzzy retraction of $\bar{L}^n \subset \bar{R}^{n+2}$ be $\bar{r} : \bar{L}^n \to \bar{L}^{n-1}$, $\bar{L}^{n-1} \subset \bar{L}^n$, and the fuzzy folding of $\bar{L}^n$ be $\bar{f} : \bar{L}^n \to \bar{L}^n$, then

(i) $\bar{f}_2 \circ \bar{r}_1 (\bar{L}^n) = \bar{r}_2 \circ \bar{f}_1 (\bar{L}^n)$

(ii) $\bar{\sigma}_{n+1} \circ (\lim_{i \to \infty} (\bar{f}_{2i} \circ \bar{r}_{2i-1} (\bar{L}^n))) (\cdots (\bar{f}_4 \circ \bar{r}_3 (\bar{f}_2 \circ \bar{r}_1 (\bar{L}^n))) \cdots ) = (\lim_{i \to \infty} (\bar{r}_{2i} \circ \bar{f}_{2i-1}) (\cdots (\bar{f}_4 \circ \bar{f}_3 (\bar{f}_2 \circ \bar{f}_1 (\bar{L}^n))) \cdots )) \circ \bar{\sigma}_1$.

Proof. (i) - First, consider the fuzzy hypercylinder coordinate of $\bar{L}^n \subset \bar{R}^{n+2}$ by $r(\eta)$ and $\theta_1(\eta), \theta_2(\eta), \ldots, \theta_{n+1}(\eta)$, one has

$\bar{x}_1 = \cos\theta_1(\eta), \bar{x}_i = \cos\theta_{i-1}(\eta), \bar{x}_i = \cos\theta_{i-1}(\eta), 1 < i < n + 1$.

Let the fuzzy retraction of the open fuzzy hypercylinder $\bar{L}^n = \{ (\bar{L}^n - \delta) \} \in \bar{f}_1 : \bar{L}^n \to \bar{L}^{n-1}$, where $\bar{x}_{n+1} = 0$, $\bar{f}_1 : \bar{L}^n \to \bar{L}^n$, the fuzzy retraction of $\bar{f}_1 (\bar{L}^n)$ is $\bar{f}_2 : \bar{f}_1 (\bar{L}^n) \to \bar{L}^{n-1}$, and the folding of $\bar{f}_1 (\bar{L}^n)$ is $\bar{f}_2 : \bar{f}_1 (\bar{L}^n) \to \bar{L}^{n-1}$.

(ii) - Let $\bar{f}_{2i} \circ \bar{r}_{2i-1}$ and $\bar{r}_{2i} \circ \bar{f}_{2i-1}$ are the compositions between the fuzzy retractions and the fuzzy foldings of $\bar{L}^n$ into itself. Also, $\bar{\sigma}_i$ is the homeomorphisms. Then

Theorem 6. If $\bar{r}_i : \exp^{-1}(\bar{L}^n) \to T_p(\bar{L}^n)$, $\bar{r}_i : \bar{L}^n \to \bar{L}^n$ and $\exp^{-1} : \bar{L}^n \to \bar{T}_p(\bar{L}^n)$. Then $\exp^{-1} \circ \bar{r}_i = \bar{r}_i \circ \exp^{-1}$.

Proof. Consider the fuzzy exponential map $\exp^{-1} : \bar{L}^n \to \bar{T}_p(\bar{L}^n)$ and the fuzzy retractions be defined as $\bar{r}_0 : \bar{L}^n \to \bar{L}_1^n, \bar{r}_1 : \bar{L}_1^n \to \bar{L}_2^n$, ..., then there are induced commutative diagrams given by
Retraction of Fuzzy Hypercylinder

\[ \tilde{L}^n \rightarrow \tilde{L}_0 \rightarrow \tilde{L}_1 \rightarrow \tilde{L}_2 \rightarrow \cdots \rightarrow \lim_{i \rightarrow \infty} (\tilde{r}_i) = \tilde{L}^{n-1} \]

\[ \exp^{-1} \]

\[ \tilde{T}_p(L^n) \rightarrow \tilde{T}_p(L^n) \rightarrow \tilde{T}_p(L^n) \rightarrow \tilde{T}_p(L^n) \rightarrow \tilde{T}_p(L^n-1) \]

\[ \lim_{i \rightarrow \infty} (\tilde{r}_i) = \tilde{T}_p(L^n-1) \]

Such that \( \exp^{-1} \circ \tilde{r}_i = \tilde{r}_i \circ \exp^{-1} \).

**Theorem 7.** The relation between the fuzzy exponential map and fuzzy folding of a fuzzy sphere \( \tilde{L}^n \) discussed from the following commutative diagram.

\[ \tilde{L}^n \rightarrow \tilde{L}_n \rightarrow \tilde{L}_n \rightarrow \tilde{L}_n \rightarrow \tilde{L}_n \rightarrow \tilde{L}_n \rightarrow \cdots \rightarrow \tilde{L}_n \]

**Proof.** Since \( \exp^{-1} : \tilde{L}^n \rightarrow \tilde{T}_p(L^n) \) be the fuzzy exponential map of \( \tilde{L}^n \), the fuzzy folding of \( \tilde{L}^n \), \( \tilde{T}_p(L^n) \) are given by \( \tilde{S} : \tilde{L}^n \rightarrow \tilde{L}^n \). Hence the following diagram is commutative, i.e. \( \exp^{-1} \circ \tilde{S} = \tilde{S} \circ \exp^{-1} \).

**Theorem 8.** Let \( \tilde{L}^n, \tilde{L}^{n-1} \) be two fuzzy hypercylinder in a fuzzy Euclidean space \( \tilde{L}^{n-1} \subset \tilde{L}^n \subset \tilde{R}^{n+2} \). Then any fuzzy retractions of \( \tilde{L}^n \) induces fuzzy retractions of \( \tilde{T}_p(L^n) \) onto \( \tilde{T}_p(L^{n-1}) \).

**Proof.** Consider two fuzzy spheres \( \tilde{L}^n, \tilde{L}^{n-1} \) immersed with a common point \( p \). Then at \( p \) the fuzzy tangent spaces will be overlapped. Now, let \( \tilde{r}_\mu : \mu(\tilde{L}^n - p) \rightarrow \mu_1(\tilde{L}^{n-1} - p) \), \( \mu_1 < \mu \), be a fuzzy retractions of the fuzzy physical character of \( \{\tilde{L}^n - p\} \) into \( \{\tilde{L}^{n-1} - p\} \). Then there is an induced fuzzy retractions \( \tilde{r}_\mu : \tilde{T}_p(\mu(\tilde{L}^n - p)) \rightarrow \tilde{T}_p(\mu_1(\tilde{L}^{n-1} - p)) \) and \( \exp^{-1} \circ \tilde{r}_\mu = \tilde{r}_\mu \circ \exp^{-1} \).

**Theorem 9.** Let \( \tilde{L}^n \subset \tilde{R}^{n+2} \), if \( \tilde{S}_T : Tp_1(L^n) \rightarrow Tp_2(L^n) \) and \( \exp((Tp_1)(L^n)) \rightarrow \tilde{L}^n_{p_i} \), then \( \exp \circ \tilde{S}_T = \tilde{S} \circ \exp \).

**Proof.** Let \( \tilde{S}_T : Tp_1(L^n) \rightarrow Tp_2(L^n), Tp_1(L^n) \) is the tangent space for \( p_1, p_2 \in Tp_1(L^n) \), either \( \mu_1 = \mu_2 \) or \( \mu_1 \neq \mu_2 \). Then for \( \tilde{S}_T : (Tp_1(L^n), \mu_1) \rightarrow (Tp_2(L^n), \mu_2) \) then either \( \tilde{S}_T(\mu_1) = (\mu_2) \) or one of these cases \( \tilde{S}_T(\mu_1) = \mu_1 \) or \( \tilde{S}_T(\max(\mu_1, \mu_2)) \). Then there is an induced fuzzy folding \( \tilde{S} : \tilde{L}^n_{p_1} \rightarrow \tilde{L}^n_{p_2} \) such that \( \tilde{S}(g_{1, \mu_1}) = (g_{2, \mu_1}) \) or \( (g_{2, \mu_2}) \) where \( g_i \) is a geodesics, \( i = 1,2 \). Also, \( \exp \circ \tilde{S}_T = \tilde{S} \circ \exp \).
In the case of the limits of fuzzy folding of $\bar{L}^n \subset \bar{R}^{\alpha+2}$ we obtain the following diagrams

\[
\begin{array}{cccccc}
\tau_1(L^n) & \tau_2(L^n) & \tau_3(L^n) & \cdots & \tau_{p_1-1}(L^n) \\
\exp & \sim \exp & \exp & \cdots & \exp \\
L^n & \bar{L}^n & \bar{L}^n & \cdots & \bar{L}^n \\
\end{array}
\]

\[
\begin{array}{cccccc}
\lim_{\alpha \to \infty} \tau_1 & \lim_{\alpha \to \infty} \tau_2 & \lim_{\alpha \to \infty} \tau_3 & \cdots & \lim_{\alpha \to \infty} \tau_{p_1-1} \\
\exp & \exp & \exp & \cdots & \exp \\
L^n & \bar{L}^n & \bar{L}^n & \cdots & \bar{L}^n \\
\end{array}
\]

Such that $\bar{\tau}_1 \circ \exp = \exp \circ \bar{\tau}_1$, also $\bar{\tau}_{\alpha} = \mu_{p_1}(\tau_{\alpha}(L^n))$ or $\mu_{p_2}(\tau_{\alpha}(L^n))$ and also $\bar{\tau}_1(\mu_1) = \mu_{p_2}(\bar{\tau}_1) = \mu_{p_2}(\bar{\tau}_1(n))$.

**Corollary 3.** Any isometric fuzzy folding and any fuzzy folding homeomorphic to this type of fuzzy folding $\bar{\tau}: \bar{L}^n \subset \bar{R}^{\alpha+2} \rightarrow \bar{L}^n \subset \bar{R}^{\alpha+2}$, there is an induced isometric fuzzy folding of the fuzzy tangent space $T_{\tau_1}(\bar{L}^n)$ such that the following diagram is commutative

\[
\begin{array}{cccccc}
(\tau_1(L^n)) & (\tau_1(L^n)) \\
\exp & \exp \\
L^n & L^n \\
\end{array}
\]

i.e. $\exp^{-1} \circ \bar{\tau}_1 = \bar{\tau}_2 \circ \exp^{-1}$

**Corollary 4.** The fuzzy deformation retract of $\bar{L}^n \subset \bar{R}^{\alpha+2}$ onto $\bar{L}^{\alpha-1} \subset (\bar{L}^n \subset \bar{R}^{\alpha+2} - (p_1, q_1))$, under the exponential map is an induced fuzzy deformation retract of $T_{\tau_1}(\bar{L}^n)$ onto $\exp^{-1}(\bar{L}^{\alpha-1}) \subset T_{\tau_1}(\bar{L}^n \subset \bar{R}^{\alpha+2} - q_1)$, which makes the diagram

\[
\begin{array}{cccccc}
(L^{\alpha-1} - p_1) & (L^{\alpha-1} - p_1) \\
\exp^{-1} & \exp^{-1} \\
\bar{L}^n - (p_1, q_1) & \bar{L}^n - (p_1, q_1) \\
\end{array}
\]

**Theorem 10.** Given the fuzzy deformation retract of $\bar{L}^n \subset \bar{R}^{\alpha+2}$ is $\bar{D}: \bar{L}^n \times I \rightarrow \bar{L}^n$, the limit of the fuzzy folding of $\bar{L}^n \times I$ is $\lim_{\alpha \to \infty} \bar{f}_m : \bar{L}^n \times I \rightarrow \bar{L}^{\alpha-1} \times I$. Then, the following diagram is commutative.

**Proof:** Let the limit of the fuzzy folding of $\left( \bar{L}^n \times I \right)$ is $\lim_{m \to \infty} \bar{f}_m : \bar{L}^n \times I \rightarrow \bar{L}^{\alpha-1} \times I$, the fuzzy deformation retract of $\bar{L}^n$ is $\bar{D}_1 : \bar{L}^n \times I \rightarrow \bar{L}^{\alpha-1}$, the limit of the fuzzy folding of $\bar{D}_1$ ($\bar{L}^n \times I$) is $\lim_{m \to \infty} \bar{f}_{m+1} : \bar{D}_1 \left( \bar{L}^n \times I \right) \rightarrow \bar{L}^{\alpha-1}$, and the fuzzy deformation retract of $\lim_{m \to \infty} \bar{f}_m \left( \bar{L}^n \times I \right)$ is $\bar{D}_2 : \lim_{m \to \infty} \bar{f}_m \left( \bar{L}^n \times I \right) \rightarrow \bar{L}^{\alpha-1}$. Hence
Retraction of Fuzzy Hypercylinder

\[ \mathcal{L}^n \times I \xrightarrow{\lim_{m \to \infty} f_m} \mathcal{L}^{n-1} \times I \]

\[ \downarrow \]

\[ \lim_{m \to \infty} f_m \mathcal{L}^{n-1} \xrightarrow{\mathcal{D}_2} \mathcal{L}^{n-2} \]

i.e. \( \mathcal{D}_2 \circ \lim_{m \to \infty} f_m (\mathcal{L}^n \times I) = \lim_{m \to \infty} f_{m+1} \mathcal{D}_1 (\mathcal{L}^n \times I) \)

**Theorem 11.** The composition of fuzzy deformation retract of fuzzy hypersphere \( \mathcal{L}^2 \subset \mathcal{R}^4 \) is a minimal retraction.

**Proof.** Now consider the following fuzzy continuous map \( \tilde{n} : \mathcal{L}^2 \times [0,1] \to \mathcal{L}^2 \), such that \( \tilde{n}(\bar{x}, s) = \bar{\beta}(\bar{x}, \frac{s}{1-s}) \), then it is easy to see that

\( \tilde{n}(\bar{x}, 0) = \bar{\beta}(\bar{x}, 0) = 0 \),

\( \tilde{n}(\bar{x}, 1) = \lim_{s \to 1} \bar{\beta}(\bar{x}, \frac{s}{1-s}) = \mathcal{L} \subset \mathcal{L}^2 \),

\( \tilde{n}(\bar{y}, s) = \bar{\beta}(\bar{y}, \frac{s}{1-s}) = \mathcal{S}^1 \subset \mathcal{L} \).

The fuzzy deformation retract of the fuzzy circle \( \mathcal{S}^1 \subset \mathcal{L} \) onto minimal fuzzy retraction \( (0,1) \) is given in polar coordinates by

\[ \tilde{r} e^{i(1-r(\mu))(\theta(\mu))}, |\theta(\mu)| \leq \frac{\pi}{2} \),

\( \tilde{\mu}(r(\mu) e^{i\theta(\mu)}) = \{ \tilde{r} e^{i(\theta(\mu)+\pi-\theta(\mu))r(\mu)}, \frac{\pi}{2} \leq \theta \leq \pi \),

\( \tilde{r} e^{i(\theta(\mu)+\pi+\theta(\mu))r(\mu)}, -\pi \leq \theta \leq -\frac{\pi}{2} \)

i.e. \( \tilde{\mu} \circ \tilde{n} \) is a fuzzy minimal retraction.

**Corollary 5.** The minimum retraction of the fuzzy hypercylinder \( \mathcal{L}^n \subset \mathcal{R}^{n+2} \) is a two chains of fuzzy elastic points up and down \( \eta_1, \eta_2 \). See Fig. (2).

**Figure 2:**
Applications

1. The stream function of the acoustic gravity tripolar vortices is generalized to permit a study of the Earth’s atmosphere under complex meteorological conditions, characterized by sheared horizontal flows and parabolic density and pressure profiles [7]. See Fig. (3).

![Figure 3:](image)

2. Consider the flow of the fluid inside a tube [3]. If we represent the velocity of the fluid as a membership degree \( \mu \in [0,1] \), then \( \mu = 1 \) in the mid of the medium where the velocity of the fluid takes a maximum and is symmetric round this line but at the edge of the tube the velocity of the fluid vanishes, i.e., \( \mu = 0 \).

3. The Ritz variational method [7] during the calculation of the ground – state energy in a fuzzy framework. Consider a Hamilton \( H \), and an arbitrary square integrable function \( \Psi \), so that \( \langle \Psi / H \Psi \rangle = 1 \). Considering \( \Psi \) as a fuzzy function and the ranking system as defined in [13], similar to [7] it can be shown that \( \langle \Psi / H \Psi \rangle \) is a fuzzy upper bound on \( E_0 \) (ground-stat energy). Now \( \langle \Psi / H \Psi \rangle \) should be minimizing the distance between \( E_0 \) and respect to a number of parameters \( (\alpha_1, \alpha_2, \ldots) \). This can be done by minimizing distance between \( E_0 \) and \( \langle \Psi / H \Psi \rangle \). The rest of the discussion is the same as that provided in [7].

Conclusion

In this paper we achieved the approval of the important of the curves and surface in n- dimensional fuzzy cylinder \( \mathbb{L}^n \subset \mathbb{R}^{n+2} \) by using some geometrical transformations. The relations between fuzzy folding, fuzzy retractions, fuzzy deformation retracts, limits of fuzzy the n- dimensional fuzzy cylinder \( \mathbb{L}^n \subset \mathbb{R}^{n+2} \) are discussed. New types of the tangent space \( T_p(L^n) \) in n- dimensional fuzzy cylinder \( \mathbb{L}^n \subset \mathbb{R}^{n+2} \) are deduced.

Acknowledgment

This project is supported by Institute of Scientific Research and Islamic Heritage Revival, Umm Al-Qura University, Saudi Arabia.
References