On Nonlocal Problems for Fractional Integro-Differential Equation in Banach Space

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Abstract

The aim of the present paper is to study the Cauchy-type problem for an integro-differential equation of fractional order with nonlocal conditions in Banach spaces. The fractional differential operator is taken in the Caputo sense. New conditions on the nonlinear terms are given to guarantee the equivalence. We shall prove the existence and uniqueness results by means of Banach fixed point and the Krasnoselskii’s fixed point theorems. At the end, an illustrative example will be introduced to justify our results.

Keywords: Nonlocal problems, Fractional integro-differential equation, fixed point theorem, Banach space

1. Introduction

Fractional differential equations are linked with extensive applications such as continuum phenomena mechanics, electrochemistry, biophysics, biotechnology engineering and so forth. For more details see studies of Kilbas et al. [16], Miller and Ross [18] Oldham and Spanier [21], Samko et al. [22], and many other references. The existence and uniqueness of solutions to fractional differential equations have attracted the attention of many scientists and researchers. For instance, (see [1, 2, 3, 4, 5, 11, 12, 13, 15, 17]). Integro-differential equations emerge in many scientific and engineering specialties,
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Oftentimes be an approximation to partial differential equations that represent a lot of the incessant phenomena, and the various a classes of fractional integro-differential equations have been taken into consideration by some authors, for more information, (see [2, 6, 7, 8, 9, 10, 19, 20, 23]). For example in [20] Momani et al. studied the local and global uniqueness results by applying Bihari's inequality and Gronwall's inequality for the Cauchy problem type

\[ \begin{align*}
{^cD}^\alpha u(t) & = f(t, u(t)) + \int_{t_0}^t K(t, s, u(s))ds, \\
u(0) & = u_0,
\end{align*} \]

where \(0 < \alpha \leq 1, f \in C([0,1] \times \mathbb{R}^n, \mathbb{R}^n)\), \(K \in C([0,1] \times [0,1] \times \mathbb{R}^n, \mathbb{R}^n)\) and \( {^cD}^\alpha \) is the Caputo fractional operator. In [7], Ahmad and Sivasundaram, considered the fractional integro-differential equation (1) with nonlocal conditions \(u(0) = u_0 - g(u)\), where \(0 < \alpha < 1, {^cD}^\alpha \) is the Caputo fractional derivative operator, \(f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}, K : [0, T] \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}\) are jointly continuous and \(g \in C([0, T], \mathbb{R})\). The authors employed Banach's contraction principle and Krasnoselskii's fixed point theorem to establish the existence and uniqueness results in Banach spaces. Wu and Liu in [23], extended the results that have been obtained in [7],[8] by employed Krasnoselskii-Krein-type conditions. Integral boundary conditions are encountered in various applications such as population dynamics, blood flow models, chemical engineering, cellular systems, underground water flow, heat transmission, plasma physics, thermoelasticity, etc. Nonlocal conditions come up when values of the function on the boundary is connected to values inside the domain. Nonlocal conditions are found to be more plausible than the standard initial conditions for the formulation of some physical phenomena in certain problems of thermodynamics, elasticity and wave propagation.

This article is concerned with the existence and uniqueness results for fractional integro-differential equations of the type

\[ \begin{align*}
{^cD}^\alpha_a u(t) & = h(u(t)) + f(t, u(t)) + \int_0^t K(t, s, u(s))ds, \\
u(a) & = \sum_{k=1}^m c_k u(\tau_k), \quad \tau_k \in (a, b),
\end{align*} \]

with nonlocal condition

where \(0 < \alpha < 1\), \( {^cD}^\alpha_a \) is the Caputo fractional derivative operator, \(f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}, K : [a, b] \times [a, b] \times \mathbb{R} \rightarrow \mathbb{R}\) are appropriate functions satisfying some conditions which will be stated later, \(\tau_k, k = 1,...,m\) are prefixed points satisfying \(a < \tau_1 \leq \cdots \leq \tau_m < b\) and \(c_k\) is real numbers.

The organization of this paper is as follows. In Section 2, we state some known notations and definitions and we also list the hypotheses which are used throughout this paper. Section 3 and Section 4 provide the proofs of the existence and uniqueness of solution to the problem (3)-(4) in Banach space. Finally, an illustrative example is presented in Section 5.

2. Preliminaries

In this section, we present some essential notations, definitions and Lemmas concerning fractional calculus and fixed point theorem. Let \(J = [a, b]\) and \(X\) is Banach space with norm \(\|\cdot\|\), by \(C(J, X), C^\alpha(J, X)\) we denotes the Banach space of all continuous bounded functions and continuously differentiable functions up to order \((n - 1)\) on \(J\), respectively. Moreover for any function \(g : J \rightarrow \mathbb{R}\), we define the norm \(\|g\|_{C(J, X)} = \max\{\|g(t)\| : t \in J\}\). In the following, the Mittag-Leffler function is given by
where \( \Gamma(.) \) is the Euler gamma function. Further, if \( 0 < \alpha < 2 \) and \( \beta > 1 \), we have \([14]\)

\[
E_{\alpha,\beta}(w) \leq \frac{1}{\alpha} w^{(-\beta)} e^{-\frac{1}{\alpha} w}.
\]

**Definition 2.1** ([16]). Let \( \alpha > 0 \) and \( g: J \to X \). The left sided Riemann--Liouville fractional integral of order \( \alpha \) of a function \( g \) is defined as

\[
I_{a^+}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-s)^{\alpha-1} g(s) ds, \quad t \in J,
\]

Provided that \( I_{a^+}^\alpha g \) exists for all \( \alpha > 0 \).

**Definition 2.2** ([16]). Let \( n \in \mathbb{N} \) and \( g \in C^n(J, X) \). The left sided Caputo fractional derivative of order \( \alpha \) of a function \( g \) is defined as

\[
cD_{a^+}^\alpha g(t) = I_{a^+}^{\alpha-n} \frac{d^n}{dt^n} g(t), \quad t \in J,
\]

where \( n = \lfloor \alpha \rfloor + 1 \), and \( \lfloor \alpha \rfloor \) denotes the integer part of the real number \( \alpha \).

**Lemma 2.3** ([16, 22]). For \( \alpha, \beta > 0 \) and \( g, p \) are appropriate functions, then for \( t \in J \), we have

1) \( I_{a^+}^\alpha I_{a^+}^\beta g(t) = I_{a^+}^{\alpha+\beta} g(t) = I_{a^+}^\beta I_{a^+}^\alpha g(t) \).

2) \( I_{a^+}^\beta (g(t) + p(t)) = I_{a^+}^\beta g(t) + I_{a^+}^\beta p(t) \).

3) \( I_{a^+}^\alpha cD_{a^+}^\nu g(t) = g(t) - g(a), \quad 0 < \alpha < 1 \).

4) \( cD_{a^+}^\nu I_{a^+}^\alpha g(t) = g(t) \).

5) \( cD_{a^+}^\nu cD_{a^+}^\nu g(t) = I_{a^+}^{\alpha-n} \frac{d^n}{dt^n} g(t), \quad 0 < \alpha < 1 \).

6) \( cD_{a^+}^\nu c = 0 \), where \( c \) is a constant.

**Lemma 2.4** Let \( \alpha > 0 \) and \( g \in C(J, X) \), then \( I_{a^+}^\alpha g \in C(J, X) \) and \( I_{a^+}^\alpha g(a) = \lim_{t \to a^+} I_{a^+}^\alpha g(t) = 0 \).

**Lemma 2.5** ([24]) (Banach fixed point theorem) Let \((U, d)\) be a non-empty complete metric space with a contraction mapping \( T: U \to U \). Then, \( T \) has a unique fixed point \( u \in U \) (i.e. \( T(u) = u, \quad u \in U \)).

**Lemma 2.6** ([24]) (Krasnoselskii fixed point theorem). Let \( E \) be bounded, closed and convex subset of a Banach space \( X \), Let \( T_1, T_2 : E \to E \) satisfying the following:

1) \( T_1x + T_2y \in E \), for every \( x, y \in E \).

2) \( T_1 \) is contraction.

3) \( T_2 \) is compact and continuous.

Then, there exists \( z \in E \) such that the equation \( z = T_1z + T_2z \) has a solution on \( E \).

### 3. Existence Result

In this section, we shall demonstrate the existence result of (3) – (4). For reader’s comfort, we list of hypotheses is supplied as follows:

(A1) \( h: C(J, X) \to X \) is continuous, bounded and there exists \( 0 < M < 1 \) such that

\[
\|h(u) - h(v)\| \leq M \|u - v\|, \quad \text{for} \quad u, v \in X.
\]

(A2) \( f: J \times X \to X \) is continuous and there exist \( \beta \in (0, 1], \quad L > 0 \) such that
\[ \|f(t, u) - f(t, v)\| \leq L\|u - v\|^\beta, \quad t \in J, \ u, v \in X. \]

(A3) \( K : D \times X \to X \) is continuous on \( D \) and there exist \( \gamma \in (0, 1], \ \rho \in L^1(J) \) such that
\[ \|K(t, u(s)) - K(t, v(s))\| \leq \rho(t)\|u - v\|, \quad (t, s) \in D, \ u, v \in X, \]

where \( D = \{(t, s) : a \leq s \leq t \leq b\} \).

First, we will state the following axiom lemma.

**Lemma 3.1** Let \( 0 < \alpha < 1 \). Assume that \( h, f \) and \( K \) are continuous functions. If \( u \in C(J, X) \) then \( u \) satisfies the problem (3) – (4) if and only if \( u \) satisfies the integral equation
\[ u(t) = \sum_{k = 1}^{m} c_k \frac{B_k}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} \left[ h(u(s)) + f(s, u(s)) + \int_s^t K(\sigma, s, u(s))d\sigma \right] ds \]
\[ + \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} \left[ h(u(s)) + f(s, u(s)) + \int_s^t K(\sigma, s, u(s))d\sigma \right] ds, \quad \text{where} \]

\[ B_k = \frac{1}{1 - \sum_{k = 1}^{m} c_k}. \]

**Proof.** Let \( u \in C(J, X) \) be a solution of (3) – (4). We show that \( u \) is also a solution of Eq. (5).

By using Lemma 2.3, we get
\[ I_a^\alpha c^\alpha D_a u(t) = u(t) - u(a). \]

Furthermore, from Eq. (3) and Definition 2.1, we have
\[ I_a^\alpha c^\alpha D_a u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} \left[ h(u(s)) + f(s, u(s)) + \int_s^t K(\sigma, s, u(s))d\sigma \right] ds. \]

Comparing Eq. (6) and Eq. (7), and using nonlocal condition Eq. (4), we obtain
\[ u(t) = \sum_{k = 1}^{m} c_k u(\tau_k) + \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} \left[ h(u(s)) + f(s, u(s)) + \int_s^t K(\sigma, s, u(s))d\sigma \right] ds. \]

Now, we substitute \( t = \tau_k \) in Eq. (8), we find that
\[ u(\tau_k) = \sum_{k = 1}^{m} c_k u(\tau_k) + \frac{1}{\Gamma(\alpha)} \int_a^{\tau_k} (\tau_k - s)^{\alpha - 1} \left[ h(u(s)) + f(s, u(s)) + \int_s^{\tau_k} K(\sigma, s, u(s))d\sigma \right] ds \]
\[ = \frac{1}{\Gamma(\alpha)} \int_a^{\tau_k} (\tau_k - s)^{\alpha - 1} \left[ h(u(s)) + f(s, u(s)) + \int_s^{\tau_k} K(\sigma, s, u(s))d\sigma \right] ds \]
\[ = \frac{B_k}{\Gamma(\alpha)} \int_a^1 (\tau_k - s)^{\alpha - 1} \left[ h(u(s)) + f(s, u(s)) + \int_s^{\tau_k} K(\sigma, s, u(s))d\sigma \right] ds. \]

Substituting Eq. (9) in Eq. (8), we get
\[ u(t) = \sum_{k = 1}^{m} c_k B_k \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} \left[ h(u(s)) + f(s, u(s)) + \int_s^t K(\sigma, s, u(s))d\sigma \right] ds \]
\[ + \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} \left[ h(u(s)) + f(s, u(s)) + \int_s^t K(\sigma, s, u(s))d\sigma \right] ds. \]

On the other hand, by applying the fractional derivative operator \( c^\alpha D_a^\alpha \) on both sides of Eq. (5) and in view of Lemma 2.3, we have
\[ c^\alpha D_a^\alpha u(t) = c^\alpha D_a^\alpha I_a c^\alpha D_a u(t) = c^\alpha D_a^\alpha h(u(t)) + c^\alpha D_a^\alpha f(t, u(t)) + \int_0^t c^\alpha D_a^\alpha K(t, s, u(s))ds \]
\[ = h(u(t)) + f(t, u(t)) + \int_0^t K(t, s, u(s))ds, \]
\[ \text{where} \quad c^\alpha D_a^\alpha f(t, u(t)) + \int_0^t c^\alpha D_a^\alpha K(t, s, u(s))ds. \]
which means that Eq.(3) holds.

Now, we show that if \( u \in C(J, X) \) satisfying Eq. (5), it also satisfies the condition Eq.(4).

Indeed, by Eq.(9), we can write

\[
\sum_{k=1}^{m} c_k B_k \frac{1}{\Gamma(\alpha)} \int_{a}^{t_k} (t_k - s)^{\alpha - 1} \left[ h(u(s)) + f(s, u(s)) + \int_{s}^{\tau_k} K(\sigma, s, u(s)) d\sigma \right] ds
\]

\[
= \sum_{k=1}^{m} c_k u(t_k) + \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t - s)^{\alpha - 1} \left[ h(u(s)) + f(s, u(s)) + \int_{s}^{t} K(\sigma, s, u(s)) d\sigma \right] ds
\]

Taking \( t \to a \) in Eq.(11), and use Lemma 2.4, we can conclude that \( u(a) = \sum_{k=1}^{m} c_k u(t_k) \). So this completes the proof.

Next, we will prove the existence of solution for the problem (3)-(4) in the space \( C(J, X) \) by means of Krasnoselski's fixed point theorem.

**Theorem 3.2.** Assume that the hypotheses \((A_1), (A_2), \) and \((A_3)\) hold. If

\[
\sum_{k=1}^{m} c_k B_k \left[ M + L + \|p\|_{L^1} \right] (t_k - a)^{\alpha} < 1. \tag{12}
\]

Then the fractional integro-differential equation (3) \( - \) (4) has a solution in \( C(J, X) \) on \( J \).

**Proof.** We transform the Cauchy problem (3) \( - \) (4) to be applicable to fixed point problem and define the operator \( F : C(J, X) \to C(J, X) \) by

\[
F u(t) = \sum_{k=1}^{m} c_k B_k \frac{1}{\Gamma(\alpha)} \int_{a}^{t_k} (t_k - s)^{\alpha - 1} \left[ h(u(s)) + f(s, u(s)) + \int_{s}^{t_k} K(\sigma, s, u(s)) d\sigma \right] ds
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t - s)^{\alpha - 1} \left[ h(u(s)) + f(s, u(s)) + \int_{s}^{t} K(\sigma, s, u(s)) d\sigma \right] ds.
\]

Before move ahead, we need to analyze the operator \( F \) into sum two operators \( P + Q \) as follows

\[
Pu(t) = \sum_{k=1}^{m} c_k B_k \frac{1}{\Gamma(\alpha)} \int_{a}^{t_k} (t_k - s)^{\alpha - 1} \left[ h(u(s)) + f(s, u(s)) + \int_{s}^{t_k} K(\sigma, s, u(s)) d\sigma \right] ds \tag{13}
\]

and

\[
Qu(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t - s)^{\alpha - 1} \left[ h(u(s)) + f(s, u(s)) + \int_{s}^{t} K(\sigma, s, u(s)) d\sigma \right] ds. \tag{14}
\]

For any function \( u \in C(J, X) \) and for some \( j \in \mathbb{N} \), we define the norm

\[
\|u\|_j = \max \{e^{-j} \|u(t)\|_c : t \in J \}. \quad \text{Note that the norms } \|u\|_j \text{ and } \|u\|_c \text{ are equivalent for } u \in C(J, X). \quad \text{Now,}
\]

we apply the Lemma 2.6 in several steps:

**Step(1):** We prove that \( P + Qu \in S_j \subset C(J, X) \), for every \( u, u^* \in S_j \).

Let

\[
\mu = \sup_{(s, u) \in D} \|f(s, u)\|, \quad \mu^* = \sup_{(s, u) \in D} \int_{s}^{t} |K(\sigma, s, u)| d\sigma, \quad \eta = \sup_{u \in S_j} \|h(u)\| \quad \text{and there exists}
\]

\[
r = \left[ \sum_{k=1}^{m} c_k B_k \left( (t_k - a)^{\alpha} + (b - a)^{\alpha} \right) \right] \frac{\eta \mu + \mu^*}{\Gamma(\alpha + 1)} + 1 \text{ such that } S_j = \{u \in C(J, X) : \|u\|_j \leq r\}.
\]

For operator \( P \): Noting the previous assumptions, then for \( u \in S_j \) and \( t \in J \), we have
\[ \|Pu(t)\| \leq \sum_{k=1}^{m} c_k B_k \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (\tau_k - s)^{\alpha-1} \times \left[ \|h(u)\| + \|f(s,u(s))\| + \int_{s}^{\tau_k} \|K(\sigma,s,u(s))\| \, d\sigma \right] \, ds \]
\[ \leq \sum_{k=1}^{m} c_k B_k \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (\tau_k - s)^{\alpha-1} \times \left[ \sup_{u \in S} \|h(u)\| + \sup_{(s,u) \in J \times S} \|f(s,u)\| + \sup_{(\sigma,s,u) \in D \times S} \int_{s}^{\tau_k} \|K(\sigma,s,u)\| \, d\sigma \right] \, ds \]
\[ \leq \sum_{k=1}^{m} c_k B_k \left[ \frac{\eta + \mu + \mu^*}{\Gamma(\alpha + 1)} \right] (\tau_k - a)^{\alpha}. \]

Thus,
\[ \|Pu\| \leq e^{-jb} \sum_{k=1}^{m} c_k B_k \left[ \frac{\eta + \mu + \mu^*}{\Gamma(\alpha + 1)} \right] (\tau_k - a)^{\alpha}. \tag{15} \]

With respect to operator \(Q\): For \(u^* \in S_r\) and \(t \in J\) with observing the preceding assumptions, we have
\[ \|Qu^*(t)\| \leq \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t - s)^{\alpha-1} \times \left[ \|h(u^*(s))\| + \|f(s,u^*(s))\| + \int_{s}^{t} \|K(\sigma,s,u^*(s))\| \, d\sigma \right] \, ds \]
\[ \leq \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t - s)^{\alpha-1} \times \left[ \sup_{u \in S} \|h(u^*)\| + \sup_{(s,u') \in J \times S} \|f(s,u')\| + \sup_{(\sigma,s,u') \in D \times S} \int_{s}^{t} \|K(\sigma,s,u^*)\| \, d\sigma \right] \, ds \]
\[ \leq \left[ \frac{\eta + \mu + \mu^*}{\Gamma(\alpha + 1)} \right] (t - a)^{\alpha}. \]
Consequently,
\[ \|Qu^*\| \leq e^{-jb} \left[ \frac{\eta + \mu + \mu^*}{\Gamma(\alpha + 1)} \right] (b - a)^{\alpha}. \tag{16} \]

From Eq.(15), Eq.(16) and definition of \(r\), we get
\[ \|Pu + Qu^*\| \leq \|Pu\| + \|Qu^*\| \]
\[ \leq e^{-jb} \sum_{k=1}^{m} c_k B_k \left[ \frac{\eta + \mu + \mu^*}{\Gamma(\alpha + 1)} \right] (\tau_k - a)^{\alpha} + e^{-jb} \left[ \frac{\eta + \mu + \mu^*}{\Gamma(\alpha + 1)} \right] (b - a)^{\alpha} \]
\[ = e^{-jb} \left[ \sum_{k=1}^{m} c_k B_k (\tau_k - a)^{\alpha} + (b - a)^{\alpha} \right] \left[ \frac{\eta + \mu + \mu^*}{\Gamma(\alpha + 1)} \right] < r. \]
This means that, \(Pu + Qu^* \in S_r\).

**Step(2):** We prove that operator \(P\) is a contraction map on \(S_r\).

Let us make \(S_r\) as in step (1), by our assumptions, then for \(u, u^* \in S_r\) and for \(t \in J\), we have
\[
\|Pz - Pz^*\| \leq \sum_{k=1}^{m} c_k B_k \left[ \frac{1}{\Gamma(\alpha)} \int_{a}^{b} (\tau_k - s)^{\alpha-1} \left( \|h(u(s)) - h(u^*(s))\| \right. \right.
\]
\[
+ \left. \|f(s, u(s)) - f(s, u^*(s))\| + \int_s^{b} K(\sigma, s, u(s)) - K(\sigma, s, u^*(s)) \|d\sigma\right) \right] ds
\]
\[
\leq \sum_{k=1}^{m} c_k B_k \left[ \frac{1}{\Gamma(\alpha)} \int_{a}^{b} (\tau_k - s)^{\alpha-1}(M\|u(s) - u^*(s)\|) \right.
\]
\[
+ L\|u(s) - u^*(s)\| + \int_s^{b} \rho(\sigma)\|u(s) - u^*(s)\| \|d\sigma\right) \right] ds
\]
\[
\leq \sum_{k=1}^{m} c_k B_k \left[ \frac{M\|u - u^*\|}{\Gamma(\alpha)} \int_{a}^{b} (\tau_k - s)^{\alpha-1}e^{\beta s} ds + \frac{L\|u - u^*\|}{\Gamma(\alpha)} \int_{a}^{b} (\tau_k - s)^{\alpha-1}e^{\beta s} ds \right.
\]
\[
+ \left. \left[ \|\rho\|_{\infty}\|u - u^*\|_{j} \int_{a}^{b} (\tau_k - s)^{\alpha-1}e^{\beta s} ds \right] \right]
\]
\[
\leq \sum_{k=1}^{m} c_k B_k \left[ M + L + \|\rho\|_{\infty} \right] \|u - u^*\|_{j}
\]
\[
= \sum_{k=1}^{m} c_k B_k \left[ M + L + \|\rho\|_{\infty} \right] e^{\alpha \tau_k} E_{1,\alpha+1}(j(\tau_k - a)) \|u - u^*\|_{j}
\]
\[
\leq \sum_{k=1}^{m} c_k B_k \left[ M + L + \|\rho\|_{\infty} \right] \left( \frac{\tau_k - a}{j} \right)^{\alpha} \|u - u^*\|_{j}.
\]

Therefore,
\[
\|Pz - Pz^*\| \leq \sum_{k=1}^{m} c_k B_k \left[ M + L + \|\rho\|_{\infty} \right] \left( \frac{\tau_k - a}{j} \right)^{\alpha} \|e^{j\tau_k}\| \|u - u^*\|_{j}.
\]

Since \( \frac{e^{j\tau_k}}{e^{j\beta}} < 1 \), and by Eq.(12), we conclude that \( P \) is contraction map on \( S_r \).

**Step(3):** We show that the operator \( Q \) is completely continuous on \( S_r \).

For this end, we consider \( S_r \) defined as in step (1) and we prove that \( (QS_r) \) is uniformly bounded, \( \overline{(QS_r)} \) is equicontinuous and \( Q : S_r \rightarrow S_r \) is continuous.

Firstly, we show that \( (QS_r) \) is uniformly bounded. Let us set \( R = \sup_{s \in J} \|f(s,0)\|, R = \|h(0)\| \) and

\[
R^* = \sup_{(\sigma,s) \in D} \|K(\sigma,s,0)\| d\sigma. \]

By relying on the previous assumptions, then for \( u \in S_r \) and \( t \in J \), we have
\[
\|Qu(t)\| \\
\leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left( \|h(u(s))-h(0)\| + \|h(0)\| \right) ds \\
+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left( \|f(s,u(s))-f(s,0)\| + \|f(s,0)\| \right) ds \\
+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \int_s^t \|K(\sigma,s,u(s))-K(\sigma,s,0)\| d\sigma ds \\
+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \int_s^t \|\sigma(s)\| d\sigma ds \\
\leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left( Me^{\alpha} \|u\|_j + R \right) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left( Le^{\alpha} \|u\|_j^\alpha + R^* \right) ds \\
+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left( \|p\|_L e^{\alpha} \|u\|_j^\alpha + R^* \right) ds \\
\leq \left( M \|u\|_j + L \|u\|_j^\alpha + \|p\|_L \|u\|_j^\alpha \right) I_{\alpha} e^{\alpha} + \frac{R + R^*}{\Gamma(\alpha + 1)} (t-a)^\alpha \\
\leq \left( Mr + Lr^\beta + \|p\|_L r^\gamma \right) e^{\alpha} (t-a)^\alpha E_{t,\alpha+1}(j(t-a)) + \frac{R + R^*}{\Gamma(\alpha + 1)} (t-a)^\alpha \\
\leq \left( Mr + Lr^\beta + \|p\|_L r^\gamma \right) \left( \frac{t-a}{j} \right)^\alpha + \frac{R + R^*}{\Gamma(\alpha + 1)} e^{\alpha} (t-a)^\alpha e^{\beta}. \\
\]

So,
\[
\|Qu\| \leq \frac{\left( Mr + Lr^\beta + \|p\|_L r^\gamma \right) + \frac{R + R^*}{\Gamma(\alpha + 1)} e^{\beta}}{j^{\alpha}} (b-a)^\alpha := \ell.
\]

This means that \( QS \subset S \), for any \( u \in S \), i.e. the set \( \{Qu : u \in S\} \) is uniformly bounded. 

Next, we will prove that \( (QS) \) is equicontinuous. 

Let for any \( u \in S \) and for each \( t_1, t_2 \in J \) with \( t_1 \leq t_2 \), we have
\[ \|Qu(t_2) - Qu(t_1)\| \]
\[ \leq \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} \left[ \|h(u(s))\| + \|f(s, u(s))\| + \int_{t_1}^{t_2} \|K(\sigma, s, u(s))\| d\sigma \right] ds \]
\[ + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1} \left[ \|h(u(s))\| + \|f(s, u(s))\| \right] ds \]
\[ + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} \int_{s}^{t_1} \|K(\sigma, s, u(s))\| d\sigma ds \]
\[ + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} \int_{s}^{t_1} \|K(\sigma, s, u(s))\| d\sigma ds \]
\[ \leq \frac{\eta + \mu}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} + \frac{\mu^*}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} (t_2 - s) ds \]
\[ + \frac{\eta + \mu}{\Gamma(\alpha)} \int_{t_1}^{t_2} ((t_2 - s)^{\alpha - 1} - (t_2 - s)^{\alpha - 1}) ds \]
\[ - \frac{\mu^*}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} (t_1 - s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} (t_2 - s) ds \]
\[ = \frac{\eta + \mu}{\Gamma(\alpha + 1)} (t_2 - t_1)^{\alpha} + \frac{\mu^*}{\Gamma(\alpha)(\alpha + 1)} (t_2 - t_1)^{\alpha + 1} \]
\[ + \frac{\eta + \mu}{\Gamma(\alpha)} ((t_1 - a)^{\alpha} + (t_2 - t_1)^{\alpha} - (t_2 - a)^{\alpha}) \]
\[ + \frac{\mu^*}{\Gamma(\alpha)(\alpha + 1)} ((t_1 - a)^{\alpha + 1} - (t_2 - t_1)^{\alpha + 1} + (t_2 - a)^{\alpha + 1}) \]
\[ = \frac{\eta + \mu}{\Gamma(\alpha + 1)} ((t_1 - a)^{\alpha} + 2(t_2 - t_1)^{\alpha} - (t_2 - a)^{\alpha}) \]
\[ + \frac{\mu^*}{\Gamma(\alpha)(\alpha + 1)} ((t_1 - a)^{\alpha + 1} - (t_2 - a)^{\alpha + 1}) \]
\[ \leq \frac{\eta + \mu}{\Gamma(\alpha + 1)} 2(t_2 - t_1)^{\alpha}. \]

where \( \eta, \mu \) and \( \mu^* \) are defined as in step (1). Observe that, the right hand side of the above inequality is independent of \( u \) and tends to zero when \( |t_2 - t_1| \to 0 \), i.e. \( \|Qu(t_2) - Qu(t_1)\| \to 0 \) which means that \((QS)\) is equicontinuous.

Finally, from the continuity of \( h, f \) and \( K \), we can directly reach that operator \( Q : S_r \to S_r \).
So, \( Q \) is relatively compact on \( S_r \). Hence, by Arzela-Ascoli theorem, the operator \( Q \) is compact on \( S_r \). An application of Lemma 2.6 shows that operator \( F = P + Q \) has a fixed point on \( S_r \). So the fractional integro-differential equation (3) – (4) has a solution \( u(t) \in C(J, X) \). This proves the Theorem 3.2.
4. Uniqueness Result

Theorem 4.1. Assume that

(B1) \( \| h(u) - h(v) \| \leq A_1 \| u - v \|, \) for \( u, v \in X. \)

(B2) \( \| f(t,u) - f(t,v) \| \leq A_2 \| u - v \|, \) \( t \in J, \ u, v \in X. \)

(B3) \( \| K(t,s,u(s)) - K(t,s,v(s)) \| \leq A_3 \| u - v \|, \) \( (t,s) \in D, \ u, v \in X. \)

If

\[ (A_1 + A_2) \leq \frac{\Gamma(\alpha+1)}{4 \left[ \sum_{k=1}^{n} c_k B_k (\tau_k - a) + (t-a)^\alpha \right]}, \]  

and

\[ A_3 \leq \frac{\Gamma(\alpha+2)}{4 \alpha \left[ \sum_{k=1}^{n} c_k B_k (\tau_k - a) + (t-a)^\alpha \right]}, \]

Then the fractional integro-differential equation (3) – (4) has a unique solution continuous on \( J. \)

Proof. Define \( F : C(J, X) \to C(J, X) \) by

\[ F(u)(t) = \sum_{k=1}^{n} c_k B_k \left( \frac{1}{\Gamma(\alpha)} \int_{a}^{\tau_k} (\tau_k - s)^{\alpha-1} \left[ h(u(s)) + f(s,u(s)) + \int_{s}^{\tau_k} K(\sigma,s,u(s))d\sigma \right] ds \right) + \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-s)^{\alpha-1} \left[ h(u(s)) + f(s,u(s)) + \int_{s}^{\tau_k} K(\sigma,s,u(s))d\sigma \right] ds, \]

and define \( B_r = \{ u \in C(J, X) : \| u \| \leq r \} \), for some \( r > 0 \), choosing

\[ r \geq 4 \left[ \sum_{k=1}^{n} c_k B_k (\tau_k - a) + (b-a)^\alpha \right] \frac{R + R^*}{\Gamma(\alpha+1)} \]

\[ + 4 \left[ \sum_{k=1}^{n} c_k B_k (\tau_k - a) + (b-a)^\alpha \right] \frac{\alpha R^*}{\Gamma(\alpha+2)}, \]

where \( R, R^* \) and \( R^* \) are defined as in Theorem 3.2. Now, we need to prove that the operator \( F \) has a fixed point on \( B_r \subset C(J, X) \). This fixed point is the unique solution of (3) – (4). In order that, we present the proof in two steps:

First step: We show that \( F B_r \subset B_r \).

By the hypotheses, then for any \( u \in B_r \) and for each \( t \in J \), we have

\[ \| (Fu)(t) \| \]

\[ \leq \sum_{k=1}^{n} c_k B_k \left( \frac{1}{\Gamma(\alpha)} \int_{a}^{\tau_k} (\tau_k - s)^{\alpha-1} \left[ \| h(u(s)) \| + \| f(s,u(s)) \| + \int_{s}^{\tau_k} \| K(\sigma,s,u(s)) \| d\sigma \right] ds \right) + \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-s)^{\alpha-1} \left[ \| h(u(s)) \| + \| f(s,u(s)) \| + \int_{s}^{\tau_k} \| K(\sigma,s,u(s)) \| d\sigma \right] ds \]
\[
\leq \sum_{k=1}^{m} c_k B_k \frac{1}{\Gamma(\alpha)} \int_{s}^{\tau_k - s} \left[ \|h(u(s)) - h(0)\| + \|h(0)\| \right] ds \\
+ \sum_{k=1}^{m} c_k B_k \frac{1}{\Gamma(\alpha)} \int_{s}^{\tau_k - s} \left[ \|f(s, u(s)) - f(s, 0)\| + \|f(s, 0)\| \right] ds \\
+ \sum_{k=1}^{m} c_k B_k \frac{1}{\Gamma(\alpha)} \int_{s}^{\tau_k - s} \int_{t}^{\tau_k} \left[ \|K(\sigma, s, u(s)) - K(\sigma, s, 0)\| + \|K(\sigma, s, 0)\| \right] d\sigma ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{s}^{\tau_k - s} \left[ \|h(u(s)) - h(0)\| + \|h(0)\| \right] ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{s}^{\tau_k - s} \left[ \|f(s, u(s)) - f(s, 0)\| + \|f(s, 0)\| \right] ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{s}^{\tau_k - s} \int_{t}^{\tau_k} \left[ \|K(\sigma, s, u(s)) - K(\sigma, s, 0)\| + \|K(\sigma, s, 0)\| \right] d\sigma ds \\
\leq \sum_{k=1}^{m} c_k B_k \tau_k - a^\alpha \left[ \frac{(A_1 + A_2)\|u\| + \bar{R} + R}{\Gamma(\alpha + 1)} + \frac{\alpha A_3\|u\| + R^*(\tau_k - a)}{\Gamma(\alpha + 2)} \right] \\
+(t-a)^\alpha \left[ \frac{(A_1 + A_2)\|u\| + \bar{R} + R}{\Gamma(\alpha + 1)} + \frac{\alpha A_3\|u\| + R^*(t-a)}{\Gamma(\alpha + 2)} \right] \\
\leq \sum_{k=1}^{m} c_k B_k (\tau_k - a^\alpha + (t-a)^\alpha) \left[ \frac{(A_1 + A_2)r}{\Gamma(\alpha + 1)} + \frac{\alpha R}{\Gamma(\alpha + 2)} \right] \\
+ \sum_{k=1}^{m} c_k B_k (\tau_k - a^\alpha + (t-a)^\alpha) \left[ \frac{\alpha R}{\Gamma(\alpha + 2)} \right] \\
\leq \sum_{k=1}^{m} c_k B_k (\tau_k - a^\alpha + (t-a)^\alpha) \left[ \frac{\bar{R} + R}{\Gamma(\alpha + 1)} \right] \\
+ \sum_{k=1}^{m} c_k B_k (\tau_k - a^\alpha + (t-a)^\alpha) \left[ \frac{\alpha R^*}{\Gamma(\alpha + 2)} \right] \\
\leq r.
\]

It follows that \( \|F u\| \leq r \), which means that \( F : B_r \to B_r \).

**Second step:** We shall show that \( F : B_r \to B_r \) is a contraction mapping.

Indeed, through the assumptions, then for any \( u, u^* \in B_r \) and for \( t \in J \), we can write
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\[ \left\| (Fu)(t) - (Fu^*)(t) \right\| \]
\[ \leq \sum_{k=1}^{m} c_k B_k \frac{1}{\Gamma(\alpha)} \int_{a}^{\tau} (\tau_k - s)^{\alpha-1} \left\| h(u(s)) - h(u^*(s)) \right\| ds \]
\[ + \sum_{k=1}^{m} c_k B_k \frac{1}{\Gamma(\alpha)} \int_{a}^{\tau} (\tau_k - s)^{\alpha-1} \left\| f(s, u(s)) - f(s, u^*(s)) \right\| ds \]
\[ + \sum_{k=1}^{m} c_k B_k \frac{1}{\Gamma(\alpha)} \int_{a}^{\tau} (\tau_k - s)^{\alpha-1} \int_{s}^{\tau} \left\| K(s, s, u(s)) - K(s, s, u^*(s)) \right\| d\sigma ds \]
\[ + \frac{1}{\Gamma(\alpha)} \int_{a}^{\tau} (t-s)^{\alpha-1} \left\| h(u(s)) - h(u^*(s)) \right\| ds \]
\[ + \frac{1}{\Gamma(\alpha)} \int_{a}^{\tau} (t-s)^{\alpha-1} \left\| f(s, u(s)) - f(s, u^*(s)) \right\| ds \]
\[ + \frac{1}{\Gamma(\alpha)} \int_{a}^{\tau} (t-s)^{\alpha-1} \int_{s}^{\tau} \left\| K(s, s, u(s)) - K(s, s, u^*(s)) \right\| d\sigma ds. \]
\[ \leq \sum_{k=1}^{m} c_k B_k (\tau_k - a)^{\alpha} \left[ \frac{A_1 + A_2}{\Gamma(\alpha+1)} + \frac{\alpha(t-a)^{\alpha}}{\Gamma(\alpha+2)} \right] \left\| u - u^* \right\| \]
\[ + (t-a)^{\alpha} \left[ \frac{A_1 + A_2}{\Gamma(\alpha+1)} + \frac{\alpha(t-a)^{\alpha}}{\Gamma(\alpha+2)} \right] \left\| u - u^* \right\| \]
\[ \leq (\Theta_1 + \Theta_2) \left\| u - u^* \right\|, \]
where
\[ \Theta_1 = \left[ \sum_{k=1}^{m} c_k B_k (\tau_k - a)^{\alpha} + (b-a)^{\alpha} \right] \frac{A_1 + A_2}{\Gamma(\alpha+1)} \]
and
\[ \Theta_2 = \left[ \sum_{k=1}^{m} c_k B_k (\tau_k - a)^{\alpha+1} + (b-a)^{\alpha+1} \right] \frac{\alpha A_3}{\Gamma(\alpha+2)}. \]

By Eq.(17) and Eq.(18), we get \((\Theta_1 + \Theta_2) \leq \frac{1}{2}\). This implies that \(F\) is contraction mapping. As a consequence of Lemma 2.5, there exists a fixed point \(u \in C(J, \mathbb{R})\) such that \(Fu = u\) which is the unique solution of (3)–(4) on \(J\). This proves the Theorem 4.1.

5. Examples

The present section contains two examples to point up the key results established in Sections 3 and 4.

**Example 5.1.** Consider the following nonlocal fractional integro-differential equation

\[ ^c D_t^{\alpha} u(t) = \frac{1}{3} \sin u(t) + \frac{[u(t)]^{\beta}}{3 + e^{-t}} + \int_{0}^{t} \frac{e^{t-s}}{2 + e^{t}}[u(s)]^{\gamma} ds, \]
\[ u(0) = \frac{1}{5} u(2). \]

Here, \(c_i = \frac{1}{2}, \tau_1 = \frac{3}{4}, \alpha = \frac{1}{2}, h(u) = \frac{1}{3} \sin(u), f(t, u) = \frac{u^\beta}{3 + e^{-t}}, \) and \(K(t, s, u) = \frac{e^{t-s}}{2 + e^{t}} u^\gamma (0 < \beta, \gamma < 1). \)

For \(u, v \in X = \mathbb{R}^+\) and \(t \in [0, 1]\). We can see that

\[ \|f(t, u) - f(t, v)\| \leq \frac{1}{4} \|u^\beta - v^\beta\| \leq \frac{1}{4} \|u - v\|^\beta, \quad 0 < \beta < 1. \]
\[ \|h(u(t)) - h(v(t))\| \leq \frac{1}{3} \|u - v\|, \]

and

\[ \|K(t, s, u(s)) - K(t, s, v(s))\| \leq \frac{e^{t}}{2 + e^{t}} \|u^{\gamma} - v^{\gamma}\| \leq \frac{1}{3} e^{t} \|u - v\|^{\gamma}, \quad 0 < \gamma < 1. \]

So, the conditions \((A1), (A2)\) and \((A3)\) are satisfied with \(M = \frac{1}{4}, L = \frac{1}{4}, \) and \(\rho(t) = \frac{1}{4} e^{t} \).

Further, it is easy to check that Eq. (12) holds too. Indeed,

\[ c_{i} B_{i} \left[ M + L + \|\rho\|_{L^1}\right] (\tau_{i} - a)^{\alpha} < 1. \]

An application of Theorem 3.2 implies that problem (20) - (21) has a solution on \([0,1]\).

**Example 5.2** Consider the following nonlocal fractional integro-differential equation

\[ \begin{aligned}
\mathcal{D}^{\frac{1}{2}}_{0^+} u(t) &= \frac{1}{24} \sin u(t) + \frac{u(t)}{44 + e^{-t}} + \int_{0}^{t} e^{-(s-t)} u(s) \frac{16}{ds}, \\
\quad u(0) &= \frac{1}{3} u\left(\frac{1}{7}\right) + \frac{1}{6} u\left(\frac{2}{7}\right). \tag{22}
\end{aligned} \]

where, \(c_{1} = \frac{1}{4}, \quad c_{2} = \frac{1}{5}, \quad (\tau_{1} = \frac{1}{4}) \leq (\tau_{2} = \frac{5}{4}), \quad \alpha = \frac{1}{2}, \quad h(u) = \frac{1}{34} \sin(u), \quad f(t, u) = \frac{u}{44 + e^{-t}}, \) and

\[ K(t, s, u) = e^{-(s-t)} u. \]

For \(u, v \in X = R^{+} \) and \(t \in [0,1]\). We can see that

\[ \|f(t, u) - f(t, v)\| \leq \frac{1}{45} \|u - v\|, \]

\[ \|h(u(t)) - h(v(t))\| \leq \frac{1}{24} \|u - v\|, \]

and

\[ \|K(t, s, u(s)) - K(t, s, v(s))\| \leq \frac{e^{-(s-t)}}{16} \|u - v\| \leq \frac{1}{16} \|u - v\|. \]

Thus, the conditions \((B1), (B2)\) and \((B3)\) are satisfied with \(A_{1} = \frac{1}{24}, A_{2} = \frac{1}{34}, \) and \(A_{3} = \frac{1}{16}. \)

Further, it is easy to check that Eq. (17) and Eq. (18) are hold too. Indeed, \(B_{1} = \frac{1}{1-c_{1}} = \frac{1}{1-4} = 2. \) Some elementary computations gives us

\[ (A_{1} + A_{2}) \leq \frac{\Gamma(\alpha + 1)}{4 \left[ c_{i} B_{i} (\tau_{1} - a)^{\alpha} + c_{2} B_{2} (\tau_{2} - a)^{\alpha} + (b - a)^{\alpha} \right]} = 0.145 \]

and

\[ A_{3} \leq \frac{\Gamma(\frac{1}{2} + 2)}{4 \alpha \left[ c_{i} B_{i} (\tau_{1} - a)^{\alpha+1} + c_{2} B_{2} (\tau_{2} - a)^{\alpha+1} + (b - a)^{\alpha+1} \right]} = 0.616. \]

Finally, by simple computations, we get \(\Theta_{1} + \Theta_{2} \leq \frac{1}{2}, \) where

\[ \Theta_{1} = \left[ \sum_{k=1}^{2} c_{k} B_{k} (\tau_{k} - a)^{\alpha} + (b - a)^{\alpha} \right] \frac{A_{1} + A_{2}}{\Gamma(\alpha + 1)} = 0.099 \]

and

\[ \Theta_{2} = \left[ \sum_{k=1}^{2} c_{k} B_{k} (\tau_{k} - a)^{\alpha+1} + (b - a)^{\alpha+1} \right] \frac{\alpha A_{3}}{\Gamma(\alpha + 2)} = 0.101. \]

By applying Theorem 4.1 the problem (22)-(23) has a unique solution on \([0,1]\).
6. Conclusions
We can conclude that the main results of this article have been successfully achieved, that is, through the Banach fixed point theorem and the Krasnoselskii fixed point theorem, extremely important results within the mathematical analysis. The existence and uniqueness of solution of the Cauchy problem type for nonlinear fractional integro-differential equation with nonlocal conditions introduced by the Caputo fractional derivative have been investigated. This paper contributes to the growth of fractional differential equations, especially in the case of fractional integro-differential equations involving a Caputo fractional derivative with nonlocal conditions.

References


